

Sharp well-posedness and ill-posedness in Fourier-Besov spaces for the viscous primitive equations of geophysics*

Jinyi Sun[†] and Shangbin Cui

Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, P. R. China

Abstract

We study well-posedness and ill-posedness for Cauchy problem of the three-dimensional viscous primitive equations describing the large scale ocean and atmosphere dynamics. By using the Littlewood-Paley analysis technique, in particular Chemin-Lerner's localization method, we prove that the Cauchy problem with Prandtl number $P = 1$ is locally well-posed in the Fourier-Besov spaces $[F\dot{B}_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3)]^4$ for $1 < p \leq \infty, 1 \leq r < \infty$ and $[F\dot{B}_{1,r}^{-1}(\mathbb{R}^3)]^4$ for $1 \leq r \leq 2$, and globally well-posed in these spaces when the initial data (u_0, θ_0) are small. We also prove that such problem is ill-posed in $[F\dot{B}_{1,r}^{-1}(\mathbb{R}^3)]^4$ for $2 < r \leq \infty$, showing that the results stated above are sharp.

Keywords: Viscous primitive equations; well-posedness; ill-posedness; Fourier-Besov spaces.

Subject Classification: 35Q35, 35Q86, 76D03.

1 Introduction

The viscous primitive equations are a fundamental mathematical model in the field of fluid geophysics. It describes the large scale ocean and atmosphere dynamics, cf. the monograph [25, 43, 45], for instance. The model reads as follows:

$$\begin{cases} \partial_t u - \nu \Delta u + \Omega e_3 \times u + (u \cdot \nabla)u + \nabla p = g\theta e_3 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \partial_t \theta - \mu \Delta \theta + (u \cdot \nabla)\theta = -\mathcal{N}^2 u_3 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \end{cases} \quad (1.1)$$

where the unknown functions $u = (u_1, u_2, u_3)$ and p denote the fluid velocity and the pressure, respectively, and θ is a scalar function representing the density fluctuation in the fluid (in the case of the ocean it depends on the temperature and the salinity, and in the case of the atmosphere it depends on the temperature), and ν, μ and g are positive constants related to viscosity, diffusivity and gravity, respectively. Moreover, Ω is the so-called Coriolis parameter, a real constant which is twice the angular velocity of the rotation around the vertical unit vector

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[†]Corresponding author. E-mails: sunjinyi333@163.com (J. Sun) and cuishb@mail.sysu.edu.cn (S. Cui).

$e_3 = (0, 0, 1)$, and \mathcal{N} is the stratification parameter, a nonnegative constant representing the Brunt-Väisälä wave frequency. The ratio $P := \frac{\nu}{\mu}$ is known as the Prandtl number and $B := \frac{\Omega}{\mathcal{N}}$ is essentially the “Burger” number of geophysics. We refer the reader to see [2, 25, 43, 45] for derivation of this model and more detailed discussions on its physical background.

If $\theta \equiv 0$, $\mathcal{N} = 0$ and $\Omega = 0$, then (1.1) reduces to the classical incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla)u = -\nabla p & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \end{cases} \quad (1.2)$$

which has drawn great attention during the past fifty more years. In 1964 Fujita and Kato [28] obtained the first result on well-posedness of the Cauchy problem of (1.2) and proved that it is locally well-posed in $H^s(\mathbb{R}^3)$ for $s \geq \frac{1}{2}$ and globally well-posed in $H^{\frac{1}{2}}(\mathbb{R}^3)$ for small initial data. These results were later extended to various other function spaces, cf. [9, 11, 24, 29, 30, 37, 38, 41, 44] and references therein. Particularly worth mentioning is that well-posedness has been established in $\dot{B}_{p,r}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ for $1 \leq p < \infty$, $1 \leq r \leq \infty$ by Cannone [9] and Planchon [44], in BMO^{-1} by Koch and Tataru [38] and in $B_{\infty,q}^{-1,\sigma}(\mathbb{R}^3)$ for $1 \leq q \leq \infty$ and $\sigma \geq 1 - \min\{1 - \frac{1}{q}, \frac{1}{q}\}$ by Cui [24]. On the other hand, the ill-posedness for Cauchy problem of (1.2) has been proved in $\dot{B}_{\infty,q}^{-1}(\mathbb{R}^3)$ for $1 \leq q \leq \infty$, in $\dot{F}_{\infty,r}^{-1}(\mathbb{R}^3)$ for $2 < r \leq \infty$ and in $B_{\infty,q}^{-1,\sigma}(\mathbb{R}^3)$ for $1 \leq q \leq \infty$ and $0 \leq \sigma < 1 - \min\{1 - \frac{1}{q}, \frac{1}{q}\}$, cf. Bourgain and Pavlović [7], Yoneda [48], Wang [47] and Cui [24], which imply that the well-posedness results obtained in [9, 44, 38, 24] are sharp.

If only $\theta \equiv 0$ and $\mathcal{N} = 0$ but $\Omega \neq 0$, then (1.1) reduces to the incompressible rotating Navier-Stokes equation

$$\begin{cases} \partial_t u - \nu \Delta u + \Omega e_3 \times u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty). \end{cases} \quad (1.3)$$

The topic of well-posedness and ill-posedness of the Cauchy problem of (1.3) have also been widely studied, cf. [2, 22, 23, 26, 27, 31, 32, 34, 35, 39, 40, 46] and the references therein. In particular, it has been proved that the Cauchy problem of (1.3) is globally well-posed for small initial data in $FM_0^{-1}(\mathbb{R}^3)$ by Giga et al. [31], in $H^{\frac{1}{2}}(\mathbb{R}^3)$ by Hieber et al. [32], in $\dot{F}B_{p,p}^{2-\frac{3}{p}}(\mathbb{R}^3)$ for $3 < p < \infty$ and $\dot{F}B_{1,1}^{-1}(\mathbb{R}^3) \cap \dot{F}B_{1,1}^0(\mathbb{R}^3)$ by Konieczny and Yoneda [40], in $\dot{F}B_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3)$ for $1 < p \leq \infty$ and $1 \leq r < \infty$ by Fang et al. [26] and in $\dot{F}B_{1,2}^{-1}(\mathbb{R}^3)$ by Iwabuchi and Takada [35]. Moreover, in [35], the ill-posedness has been verified in $\dot{F}B_{1,r}^{-1}(\mathbb{R}^3)$ for $2 < r \leq \infty$, which implies that the well-posedness results in $\dot{F}B_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3)$ for $1 < p \leq \infty$, $1 \leq r < \infty$ and $\dot{F}B_{1,2}^{-1}(\mathbb{R}^3)$ obtained in [26] and [35] are sharp.

In this paper we study well-posedness and ill-posedness of the Cauchy problem of (1.1):

$$\begin{cases} \partial_t u - \nu \Delta u + \Omega e_3 \times u + (u \cdot \nabla)u + \nabla p = g\theta e_3 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \partial_t \theta - \mu \Delta \theta + (u \cdot \nabla)\theta = -\mathcal{N}^2 u_3 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.4)$$

where u_0 and θ_0 are given initial functions. A short review of existing work on this topic is as follows. In [2], by taking full advantage of the absence of resonances between the fast rotation and the nonlinear advection, Babin, Maholov and Nicolaenko obtained global well-posedness of problem (1.4) in $[H^s(\mathbb{T}^3)]^4$ with $s \geq 3/4$ for small initial data when the stratification parameter \mathcal{N} is sufficiently large. Later on, by constructing the solution of a quasi-geostrophic system related to equations (1.1) and using some Strichartz-type estimates, Charve [15] verified global well-posedness of problem (1.4) in $[\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)]^4$ for arbitrary (i.e., not necessarily small) initial data under the assumptions that both Ω and \mathcal{N} are sufficiently large (depending on the scale of the initial data). In [18], Charve further considered global well-posedness of (1.4) in less regular initial value spaces. We also mention the interesting work of Charve and Ngo [20] on well-posedness of the problem (1.4) with anisotropic viscosities. Recently, Koba, Mahalov and Yoneda [36] proved global well-posedness of problem (1.4) for any given $(u_0, \theta_0) \in [\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)]^4$ with $\partial_2 u_0^1 - \partial_1 u_0^2 = 0$ in the special case Prandtl number $P = 1$ provided either condition (a): the absolute value of “Burger” number $|B| < \sqrt{g}$ and \mathcal{N} is sufficiently large (depending on the scale of the initial data) or condition (b): the absolute value of “Burger” number $|B| > \sqrt{g}$ and both Ω and \mathcal{N} are sufficiently large (depending on the scale of initial data) holds. They also obtained the global well-posedness of problem (1.4) for uniformly small initial data in $[\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)]^4$. For the other related studies on the viscous primitive equations (1.1), we refer the interested reader to [1, 3, 4, 12, 13, 14, 16, 17, 19, 21].

From the above review we see that concerning the well-posedness issue, there is a big gap between the standard Navier-Stokes initial value problem (1.2) and the initial value problem (1.4) of the primitive equations: We know that the problem (1.2) is well-posed in a lot of function spaces of negative regularity indices, so that it has at least local-in-time solutions for a large group of rough initial data. For the problem (1.4), however, existing results only show that it is locally well-posed in some function spaces of the regularity index $s \geq \frac{1}{2}$. It is natural to ask whether similar results concerning well-posedness and ill-posedness of the initial value problem in function spaces of negative regularity indices as for the Navier-Stokes equations can be established for the primitive equations. This is the main motivation of the present work. However, due to the influence of the oscillations caused by the rotation (i.e., the term $\Omega e_3 \times u$) and the stratification (i.e., the terms $g\theta e_3$ and $\mathcal{N}^2 u_3$), a big portion of the integral estimates for the Stokes semigroup $\{e^{t\Delta}\}_{t \geq 0}$ (which relates to the Navier-Stokes equations) do not work for the Stokes-Coriolis-Stratification semigroup $\{T_{\Omega, \mathcal{N}}(t)\}_{t \geq 0}$ (see Section 2.1 for the definition) related to the primitive equations. Consequently, the usual function spaces used in the study of the Navier-Stokes equations such as the homogeneous and inhomogeneous Besov spaces $\dot{B}_{pr}^s(\mathbb{R}^3)$ and $B_{pr}^s(\mathbb{R}^3)$ and the space $BMO^{-1}(\mathbb{R}^3)$ are not suitable for the primitive equations. In this work, as in [26, 35, 40], we use the Fourier-Besov spaces $[FB_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3)]^4$ ($1 \leq p, r \leq \infty$) as the initial value space to study well-posedness issue of the primitive equations.

To give the precise statements of our main results, we first recall the definitions of the homogeneous Besov spaces $\dot{B}_{p,r}^s(\mathbb{R}^3)$ and the Fourier-Besov space $FB_{p,r}^s(\mathbb{R}^3)$. As usual we denote by $\mathcal{S}(\mathbb{R}^3)$ the space of Schwartz functions on \mathbb{R}^3 , and by $\mathcal{S}'(\mathbb{R}^3)$ the space of tempered distributions on \mathbb{R}^3 . Choose two radial function $\varphi, \psi \in \mathcal{S}(\mathbb{R}^3)$ such that their Fourier transform

$\hat{\varphi}$ and $\hat{\psi}$ satisfies the following properties:

$$\text{supp } \hat{\varphi} \subset \mathcal{B} := \{\xi \in \mathbb{R}^3 : |\xi| \leq \frac{4}{3}\},$$

$$\text{supp } \hat{\psi} \subset \mathcal{C} := \{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\},$$

and, furthermore,

$$\sum_{j \in \mathbb{Z}} \hat{\psi}(2^{-j}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Let $\varphi_j(x) := 2^{3j}\varphi(2^j x)$ and $\psi_j(x) := 2^{3j}\psi(2^j x)$ for all $j \in \mathbb{Z}$. We define by Δ_j and S_j the following operators in $\mathcal{S}'(\mathbb{R}^3)$:

$$\Delta_j f := \psi_j * f \quad \text{and} \quad S_j f := \varphi_j * f \quad \text{for } j \in \mathbb{Z} \quad \text{and} \quad f \in \mathcal{S}'(\mathbb{R}^3).$$

Define $\mathcal{S}'_h(\mathbb{R}^3) := \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}[\mathbb{R}^3]$, where $\mathcal{P}[\mathbb{R}^3]$ denotes the linear space of polynomials on \mathbb{R}^3 . It is known that there hold the following decompositions

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f \quad \text{and} \quad S_j f = \sum_{j' \leq j-1} \Delta_{j'} f \quad \text{in } \mathcal{S}'_h(\mathbb{R}^3),$$

see [5] for reference. With our choice of φ and ψ , it is easy to verify that

$$\Delta_j \Delta_k f = 0 \quad \text{if } |j - k| \geq 2 \quad \text{and}$$

$$\Delta_j (S_{k-1} f \Delta_k f) = 0 \quad \text{if } |j - k| \geq 5.$$

Definition 1.1 ([5]) For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the homogeneous Besov spaces $\dot{B}_{p,r}^s(\mathbb{R}^3)$ consists of those distributions u in $\mathcal{S}'_h(\mathbb{R}^3)$ such that

$$\|u\|_{\dot{B}_{p,r}^s} := \left\| \left\{ 2^{js} \|\Delta_j u\|_{L^p} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

Definition 1.2 ([35, 40]) For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the Fourier-Besov space $\dot{F}B_{p,r}^s(\mathbb{R}^3)$ is defined to be the set of all tempered distributions $u \in \mathcal{S}'_h(\mathbb{R}^3)$ such that

$$\|u\|_{\dot{F}B_{p,r}^s} := \left\| \left\{ 2^{js} \|\widehat{\Delta_j u}\|_{L^p} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

Fourier-Besov spaces are introduced in the literature very recently, and sometime they are entitled with other different names. An early paper by Cannone and Karch [10] studied well-posedness of the Cauchy problem of (1.2) in the space \mathcal{PM}^2 , which is in fact the space $\dot{F}B_{\infty,\infty}^2(\mathbb{R}^3)$. Iwabuchi [33] investigated well-posedness and ill-posedness for Cauchy problem of Keller-Segel system in $\dot{B}_q^{-2}(2 \leq q \leq \infty)$, which is in fact the space $\dot{F}B_{1,q}^{-2}(\mathbb{R}^3)(2 \leq q \leq \infty)$. Lei and Lin [41] proved global existence of mild solutions to the Cauchy problem of (1.2) in \mathcal{X}^{-1} , which is in fact equal to the space $\dot{F}B_{1,1}^{-1}(\mathbb{R}^3)$. Cannone and Wu [11] extended the result in [41] to the Fourier-Herz space $\dot{\mathcal{B}}_q^{-1}(1 \leq q \leq 2)$, which is in fact the space $\dot{F}B_{1,q}^{-1}(\mathbb{R}^3)(1 \leq q \leq 2)$,

and Liu and Zhao [42] considered the global well-posedness of the Cauchy problem of generalized magneto-hydrodynamic equations in the Fourier-Herz spaces $\dot{\mathcal{H}}_q^{-(2\beta-1)}$ ($1 \leq q \leq 2$), which is in fact the space $\dot{F}B_{1,q}^{-(2\beta-1)}(\mathbb{R}^3)$ ($1 \leq q \leq 2$). Systematic utilization of the Fourier-Besov spaces $\dot{F}B_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3)$ first appeared in the references [35, 40] mentioned above.

Note that the definition of the Fourier-Besov spaces $\dot{F}B_{p,r}^s(\mathbb{R}^3)$ has some similar feature with that of the classical homogeneous Besov spaces $\dot{B}_{p,r}^s(\mathbb{R}^3)$: they both measure regularity of a function u with index s which depicts the decay or increment speed of its Fourier transform $\hat{u}(\xi)$ as $\xi \rightarrow \infty$ via dyadic decomposition. The only difference in their definitions is that for the Fourier-Besov spaces such measurement is made purely in the frequency space, while for the Besov spaces $\dot{B}_{p,r}^s(\mathbb{R}^3)$ this is done in both frequency and physical spaces jointly. Precisely because in the definition of the Fourier-Besov spaces regularity of a function is measured purely through its frequencies, they are very useful in the study of partial differential equations which are not of purely dissipative type but instead of dissipative and dispersive joint type, such as the equations (1.1) and (1.3). Relations between the Fourier-Besov spaces $\dot{F}B_{p,r}^s(\mathbb{R}^3)$ and the homogeneous Besov spaces $\dot{B}_{p,r}^s(\mathbb{R}^3)$ are as follows:

$$\dot{F}B_{2,r}^s(\mathbb{R}^3) = \dot{B}_{2,r}^s(\mathbb{R}^3) \quad \text{for } s \in \mathbb{R} \text{ and } r \in [1, \infty]$$

and

$$\dot{F}B_{p,r}^s(\mathbb{R}^3) \hookrightarrow \dot{B}_{p',r}^s(\mathbb{R}^3) \quad \text{and} \quad \dot{B}_{p,r}^s(\mathbb{R}^3) \hookrightarrow \dot{F}B_{p',r}^s(\mathbb{R}^3)$$

for $s \in \mathbb{R}$, $p \in [1, 2]$ and $r \in [1, \infty]$. These relations can be easily proved by using the Plancherel identity and the Hausdorff-Young inequality, cf. [35, 40].

Definition 1.3 For $T > 0$, $s \in \mathbb{R}$ and $1 \leq r, \delta \leq \infty$, the Chemin-Lerner type space $\tilde{L}^\delta(0, T; \dot{F}B_{p,r}^s(\mathbb{R}^3))$ built on $\dot{F}B_{p,r}^s(\mathbb{R}^3)$ is defined to be the set of all strongly measurable functions $u : (0, T) \rightarrow \dot{F}B_{p,r}^s(\mathbb{R}^3)$ such that

$$\|u\|_{\tilde{L}^\delta(0, T; \dot{F}B_{p,r}^s)} := \left\| \left\{ 2^{js} \|\widehat{\Delta_j u}\|_{L^\delta(0, T; L^p)} \right\}_{j \in \mathbb{Z}} \right\|_{l^r(\mathbb{Z})} < \infty.$$

The main results of this paper are the following three theorems:

Theorem 1.4 Let Prandtl number $P = 1$, i.e., $\nu = \mu$. Assume that $p \in (1, \infty]$ and $r \in [1, \infty)$. Then for any $(u_0, \theta_0) \in [\dot{F}B_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3)]^4$ satisfying $\operatorname{div} u_0 = 0$, there exists corresponding $T > 0$ such that problem (1.4) possesses a unique mild solution $(u, \theta) \in [C([0, T], \dot{F}B_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3))]^4 \cap X_T^\alpha$, where

$$X_T^\alpha := [\tilde{L}^{\frac{2}{1+\alpha}}(0, T; \dot{F}B_{p,r}^{3-\frac{3}{p}+\alpha}(\mathbb{R}^3))]^4 \cap [\tilde{L}^{\frac{2}{1-\alpha}}(0, T; \dot{F}B_{p,r}^{3-\frac{3}{p}-\alpha}(\mathbb{R}^3))]^4$$

(for an arbitrarily chosen but fixed number $\alpha \in (0, 1)$). Moreover, there exists a constant $C > 0$ independent of ν , Ω and \mathcal{N} such that if

$$\|(u_0, \sqrt{g}\theta_0/\mathcal{N})\|_{\dot{F}B_{p,r}^{2-\frac{3}{p}}} \leq C\nu, \tag{1.5}$$

then problem (1.4) possesses a unique global mild solution in the class $[C([0, \infty); \dot{F}B_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3))]^4 \cap X_\infty^\alpha$.

Theorem 1.5 Let Prandtl number $P = 1$, i.e., $\nu = \mu$, and $r \in [1, 2]$. Then for any $(u_0, \theta_0) \in [\dot{F}B_{1,r}^{-1}(\mathbb{R}^3)]^4$ satisfying $\operatorname{div} u_0 = 0$, there exists corresponding $T > 0$ such that problem (1.4) possesses a unique mild solution $(u, \theta) \in [C([0, T], \dot{F}B_{1,r}^{-1}(\mathbb{R}^3))]^4 \cap Y_T^\alpha$, where

$$Y_T^\alpha := [\tilde{L}^{\frac{2}{1+\alpha}}(0, T; \dot{F}B_{1,r}^\alpha(\mathbb{R}^3))]^4 \cap [\tilde{L}^{\frac{2}{1-\alpha}}(0, T; \dot{F}B_{1,r}^{-\alpha}(\mathbb{R}^3))]^4$$

(for an arbitrarily chosen but fixed number $\alpha \in (0, 1)$). Moreover, there exists a constant $C > 0$ independent of ν , Ω and \mathcal{N} such that if

$$\|(u_0, \sqrt{g}\theta_0/\mathcal{N})\|_{\dot{F}B_{1,r}^{-1}} \leq C\nu, \quad (1.6)$$

then problem (1.4) possesses a unique global mild solution in the class $[C([0, \infty); \dot{F}B_{1,r}^{-1}(\mathbb{R}^3))]^4 \cap Y_\infty^\alpha$.

Theorem 1.6 Let Prandtl number $P = 1$, i.e., $\nu = \mu$ and $2 < r \leq \infty$. Then the problem (1.4) is ill-posed in $[\dot{F}B_{1,r}^{-1}(\mathbb{R}^3)]^4$ in the sense that the solution map $(u_0, \theta_0) \mapsto (u, \theta)$ from $[\dot{F}B_{1,r}^{-1}(\mathbb{R}^3)]^4$ to $[C([0, T], \dot{F}B_{1,r}^{-1}(\mathbb{R}^3))]^4$, if exists, is not continuous at $(u_0, \theta_0) = (0, 0)$.

Theorems 1.4 and 1.5 show that the Cauchy problem (1.4) with Prandtl number $P = 1$ is locally well-posed in the Fourier-Besov spaces $[\dot{F}B_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3)]^4$ for $1 < p \leq \infty, 1 \leq r < \infty$ and $p = 1, 1 \leq r \leq 2$, and globally well-posed in these spaces when the initial data (u_0, θ_0) satisfy smallness conditions (1.5) and (1.6). On the other hand, Theorem 1.6 tells us that this problem is ill-posed in $[\dot{F}B_{1,r}^{-1}(\mathbb{R}^3)]^4$ for $2 < r \leq \infty$. Hence, we have established the sharp well-posedness and ill-posedness in Fourier-Besov spaces $[\dot{F}B_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3)]^4$ for problem (1.4) under the special case Prandtl number $P = 1$. The condition $P = 1$ is imposed for similar technical reasons as in [36]; to the present we are not clear whether this condition can be removed or not.

The proofs of Theorems 1.4 and 1.5 use the standard Picard iteration argument. The main point is that in order to get the sharp well-posedness results as stated in these theorems we must be very careful in the construction of the iteration scheme. Indeed, as well known, to get well-posedness of Cauchy problems of evolution equations in function spaces possessing sufficiently high regularity, there are many different choices of the iteration scheme. However, most of those schemes do not work in function spaces of low regularity. Concerning the problem (1.4), we shall use the same iteration scheme as that used in the literature [36]. Thanks to the hypothesis $P = 1$ or $\mu = \nu$, the semigroup $T_{\Omega, N}$ related to that iteration scheme satisfies certain very nice estimates similar to those established for the semigroup related to the equations (1.3) in the literature [35]; see Section 2 for details. The proof of Theorem 1.6 is much harder. We shall use some arguments similar to those in [6] and [35] to prove this theorem. Note that since in the present case we consider the stratification effects of the flow and one more unknown function θ than in [35], the analysis is more involved; see Section 3 for details.

The rest of this paper is organized as follows. In Section 2 we first transform the initial value problem (1.4) into an equivalent integral equation (which is essentially the same as to construct an iteration scheme), next establish some linear estimates and product laws, and finally we present the proofs of Theorems 1.4 and 1.5. The last section is devoted to giving the proof of Theorem 1.6.

Throughout this paper, we shall use C and c to denote universal constants whose value may change from line to line. Both $\mathcal{F}g$ and \hat{g} stand for Fourier transform of g with respect to space variable, while \mathcal{F}^{-1} stands for the inverse Fourier transform. Besides, since we only consider the case $P = 1$, we always assume that $\mu = \nu$.

2 Proofs of Theorems 1.4 and 1.5

In this section we present the proofs of Theorems 1.4 and 1.5. To this end, we first transform the Cauchy problem (1.4) into an equivalent integral equation, and next use the Littlewood-Paley analysis technique to establish some linear estimates and product laws.

2.1 Rewriting (1.4) into an integral equation

By setting $N := \mathcal{N}\sqrt{g}$, $v := (v^1, v^2, v^3, v^4) := (u^1, u^2, u^3, \sqrt{g}\theta/\mathcal{N})$, $v_0 := (v_0^1, v_0^2, v_0^3, v_0^4) := (u_0^1, u_0^2, u_0^3, \sqrt{g}\theta_0/\mathcal{N})$ and $\tilde{\nabla} := (\partial_1, \partial_2, \partial_3, 0)$, (1.4) can be rewritten into the following problem

$$\begin{cases} \partial_t v + \mathcal{A}v + \mathcal{B}v + \tilde{\nabla}p = -(v \cdot \tilde{\nabla})v & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \tilde{\nabla} \cdot v = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ v|_{t=0} = v_0 & \text{in } \mathbb{R}^3, \end{cases} \quad (2.1)$$

where

$$\mathcal{A} := \begin{pmatrix} -\nu\Delta & 0 & 0 & 0 \\ 0 & -\nu\Delta & 0 & 0 \\ 0 & 0 & -\nu\Delta & 0 \\ 0 & 0 & 0 & -\mu\Delta \end{pmatrix} \quad \text{and} \quad \mathcal{B} := \begin{pmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -N \\ 0 & 0 & N & 0 \end{pmatrix}. \quad (2.2)$$

Lemma 3.3 in [36], together with the fact $e^{(\mathcal{A}+\mathcal{B})t} = e^{\mathcal{A}t}e^{\mathcal{B}t}$ for $\nu = \mu$, gives the explicitly expression of Stokes-Coriolis-Stratification semigroup $T_{\Omega,N}(t)(t \geq 0)$ corresponding to the linear problem of (2.1) via Fourier transform

$$T_{\Omega,N}(t)f := \mathcal{F}^{-1} \left[\cos\left(\frac{|\xi|'}{|\xi|}t\right) e^{-\nu|\xi|^2 t} M_1 \hat{f} + \sin\left(\frac{|\xi|'}{|\xi|}t\right) e^{-\nu|\xi|^2 t} M_2 \hat{f} + e^{-\nu|\xi|^2 t} M_3 \hat{f} \right], \quad (2.3)$$

where

$$|\xi| := \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}, \quad |\xi|' := |\xi|'_{\Omega,N} := \sqrt{N^2 \xi_1^2 + N^2 \xi_2^2 + \Omega^2 \xi_3^2} \quad (2.4)$$

for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$,

$$M_1 = \begin{pmatrix} \frac{\Omega^2 \xi_3^2}{|\xi|'^2} & 0 & -\frac{N^2 \xi_1 \xi_3}{|\xi|'^2} & \frac{\Omega N \xi_2 \xi_3}{|\xi|'^2} \\ 0 & \frac{\Omega^2 \xi_3^2}{|\xi|'^2} & -\frac{N^2 \xi_2 \xi_3}{|\xi|'^2} & -\frac{\Omega N \xi_1 \xi_3}{|\xi|'^2} \\ -\frac{\Omega^2 \xi_1 \xi_3}{|\xi|'^2} & -\frac{\Omega^2 \xi_2 \xi_3}{|\xi|'^2} & \frac{N^2 (\xi_1^2 + \xi_2^2)}{|\xi|'^2} & 0 \\ \frac{\Omega N \xi_2 \xi_3}{|\xi|'^2} & -\frac{\Omega N \xi_1 \xi_3}{|\xi|'^2} & 0 & \frac{N^2 (\xi_1^2 + \xi_2^2)}{|\xi|'^2} \end{pmatrix}, \quad (2.5)$$

$$M_2 = \begin{pmatrix} 0 & -\frac{\Omega \xi_3^2}{|\xi| |\xi|'} & \frac{\Omega \xi_2 \xi_3}{|\xi| |\xi|'} & \frac{N \xi_1 \xi_3}{|\xi| |\xi|'} \\ \frac{\Omega \xi_3^2}{|\xi| |\xi|'} & 0 & -\frac{\Omega \xi_1 \xi_3}{|\xi| |\xi|'} & \frac{N \xi_2 \xi_3}{|\xi| |\xi|'} \\ -\frac{\Omega \xi_2 \xi_3}{|\xi| |\xi|'} & \frac{\Omega \xi_1 \xi_3}{|\xi| |\xi|'} & 0 & -\frac{N (\xi_1^2 + \xi_2^2)}{|\xi| |\xi|'} \\ -\frac{N \xi_1 \xi_3}{|\xi| |\xi|'} & -\frac{N \xi_2 \xi_3}{|\xi| |\xi|'} & \frac{N (\xi_1^2 + \xi_2^2)}{|\xi| |\xi|'} & 0 \end{pmatrix} \quad (2.6)$$

and

$$M_3 = \begin{pmatrix} \frac{N^2 \xi_2^2}{|\xi|'^2} & -\frac{N^2 \xi_1 \xi_2}{|\xi|'^2} & 0 & -\frac{N \Omega \xi_2 \xi_3}{|\xi|'^2} \\ -\frac{N^2 \xi_1 \xi_2}{|\xi|'^2} & \frac{N^2 \xi_1^2}{|\xi|'^2} & 0 & \frac{N \Omega \xi_1 \xi_3}{|\xi|'^2} \\ 0 & 0 & 0 & 0 \\ -\frac{N \Omega \xi_2 \xi_3}{|\xi|'^2} & \frac{N \Omega \xi_1 \xi_3}{|\xi|'^2} & 0 & \frac{\Omega^2 \xi_3^2}{|\xi|'^2} \end{pmatrix}. \quad (2.7)$$

Note that, denoting by $M_{jk}^l(\xi)$ the (j, k) -th component of the matrix $M_l(\xi)$, it is obvious that non-vanishing $M_{jk}^l(\xi)$ satisfies

$$|M_{jk}^l(\xi)| \leq 2 \quad \text{for } \xi \in \mathbb{R}^3, \quad j, k = 1, 2, 3, 4, \quad l = 1, 2, 3.$$

Define the partial Helmholtz projection operator $\tilde{\mathbb{P}} = (\tilde{\mathbb{P}}_{jk})_{4 \times 4}$ by

$$\tilde{\mathbb{P}}_{jk} := \begin{cases} \delta_{jk} + R_j R_k, & 1 \leq j, k \leq 3, \\ \delta_{jk}, & \text{otherwise,} \end{cases} \quad (2.8)$$

where δ_{jk} is the Kronecker's delta notation and R_j ($j = 1, 2, 3$) are the Riesz transforms on \mathbb{R}^3 . Then, by using the Duhamel principle, we easily see that problem (2.1) is equivalent to the following integral equation

$$v(t) = T_{\Omega, N}(t)v_0 - \int_0^t T_{\Omega, N}(t - \tau) \tilde{\mathbb{P}} \tilde{\nabla} \cdot [v(\tau) \otimes v(\tau)] d\tau. \quad (2.9)$$

2.2 Linear estimates and product laws

Next we establish some basic estimates which will play a crucial role in the proofs of Theorems 1.4 and 1.5. We first consider linear estimates for the semigroup $\{T_{\Omega, N}(t)\}_{t \geq 0}$.

Lemma 2.1 Let $T > 0$, $s \in \mathbb{R}$, $p, r \in [1, \infty]$ and $\alpha \in [0, 1]$. There exists a constant $C > 0$ such that

$$\|T_{\Omega, N}(\cdot)u_0\|_{\dot{L}^{\frac{2}{1 \pm \alpha}}(0, T; F_{p, r}^{s+1 \pm \alpha})} \leq C \nu^{-\frac{(1 \pm \alpha)}{2}} \|u_0\|_{F_{p, r}^s}$$

for $u_0 \in \dot{F}B_{p,r}^s(\mathbb{R}^3)$.

Proof. Since $\text{supp} \hat{\psi}_j \subset \{\xi \in \mathbb{R}^3 : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, one has

$$\|\mathcal{F}[\Delta_j T_{\Omega,N}(\cdot) u_0]\|_{L^p} \leq C \left\{ \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} e^{-\nu|\xi|^{2p}} |\hat{\psi}_j(\xi) \hat{u}_0(\xi)|^p d\xi \right\}^{\frac{1}{p}} \leq C e^{-\nu 2^{2j} t} \|\hat{\psi}_j \hat{u}_0\|_{L^p}$$

for all $t \geq 0$, which yields that

$$\|\mathcal{F}[\Delta_j T_{\Omega,N}(\cdot) u_0]\|_{L^{\frac{2}{1 \pm \alpha}}(0,T;L^p)} \leq C \left(\frac{1 - e^{\nu 2^{2j} \frac{2}{1 \pm \alpha} T}}{\nu 2^{2j} \frac{2}{1 \pm \alpha}} \right)^{\frac{1 \pm \alpha}{2}} \|\hat{\psi}_j \hat{u}_0\|_{L^p}.$$

Thus, we have

$$\begin{aligned} \|T_{\Omega,N}(\cdot) u_0\|_{\tilde{L}^{\frac{2}{1 \pm \alpha}}(0,T;\dot{F}B_{p,r}^{s+1 \pm \alpha})} &\leq C \left[\sum_{j \in \mathbb{Z}} \left(\frac{1 \pm \alpha}{2\nu} \right)^{\frac{(1 \pm \alpha)r}{2}} (2^{js} \|\hat{\psi}_j \hat{u}_0\|_{L^p})^r \right]^{\frac{1}{r}} \\ &\leq C \nu^{-\frac{(1 \pm \alpha)}{2}} \|u_0\|_{\dot{F}B_{p,r}^s}. \end{aligned}$$

□

Lemma 2.2 Let $T > 0$, $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$ and $\alpha \in [0, 1]$. There exists a constant $C > 0$ such that

$$\left\| \int_0^t T_{\Omega,N}(t - \tau) f(\tau) d\tau \right\|_{\tilde{L}^{\frac{2}{1 \pm \alpha}}(0,T;\dot{F}B_{p,r}^{s+1 \pm \alpha})} \leq C \nu^{-(1 + \frac{1 \pm \alpha}{2} - \frac{1}{\rho})} \|f\|_{\tilde{L}^\rho(0,T;\dot{F}B_{p,r}^{s-2 + \frac{2}{\rho}})}$$

for $f \in \tilde{L}^\rho(0,T;\dot{F}B_{p,r}^{s-2 + \frac{2}{\rho}}(\mathbb{R}^3))$ with $\rho \in [1, \frac{2}{1 \pm \alpha}]$.

Proof. By the definition of the $\tilde{L}^{\frac{2}{1 \pm \alpha}}(0,T;\dot{F}B_{p,r}^{s+1 \pm \alpha}(\mathbb{R}^3))$ and by Young's inequality, one has

$$\begin{aligned} &\left\| \int_0^t T_{\Omega,N}(t - \tau) f(\tau) d\tau \right\|_{\tilde{L}^{\frac{2}{1 \pm \alpha}}(0,T;\dot{F}B_{p,r}^{s+1 \pm \alpha})} \\ &\leq C \left\| \left\{ 2^{j(s+1 \pm \alpha)} \int_0^t e^{-\nu(t-\tau)2^{2j}} \|\hat{f}(\tau) \cdot \hat{\psi}_j\|_{L^p} d\tau \right\}_{r \in \mathbb{Z}} \right\|_{L^{\frac{2}{1 \pm \alpha}}(0,T)} \\ &\leq C \left\| \left\{ 2^{j(s+1 \pm \alpha)} \|e^{-\nu t 2^{2j}}\|_{L^m(0,T)} \|\hat{f}(\tau) \cdot \hat{\psi}_j\|_{L^\rho(0,T;L^p)} \right\}_{r \in \mathbb{Z}} \right\|_{l^r(\mathbb{Z})}, \end{aligned}$$

where $1 + \frac{1 \pm \alpha}{2} = \frac{1}{\rho} + \frac{1}{m}$. Thus, we obtain

$$\begin{aligned} &\left\| \int_0^t T_{\Omega,N}(t - \tau) f(\tau) d\tau \right\|_{\tilde{L}^{\frac{2}{1 \pm \alpha}}(0,T;\dot{F}B_{p,r}^{s+1 \pm \alpha})} \\ &\leq C \nu^{-(1 + \frac{1 \pm \alpha}{2} - \frac{1}{\rho})} \left(\sum_{j \in \mathbb{Z}} 2^{j(s+1 \pm \alpha)r} 2^{-2j(1 + \frac{1 \pm \alpha}{2} - \frac{1}{\rho})r} \|\hat{f}(\tau) \cdot \hat{\psi}_j\|_{L^\rho(0,T;L^p)}^r \right)^{\frac{1}{r}} \\ &\leq C \nu^{-(1 + \frac{1 \pm \alpha}{2} - \frac{1}{\rho})} \|f\|_{\tilde{L}^\rho(0,T;\dot{F}B_{p,r}^{s-2 + \frac{2}{\rho}})}. \end{aligned}$$

□

We now turn to establish product laws.

Lemma 2.3 Let $T > 0$, $\alpha \in (0, 1]$, $p \in (1, \infty]$ and $r \in [1, \infty]$. There exists a constant $C > 0$ such that

$$\begin{aligned} \|fg\|_{\tilde{L}^1(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}})} &\leq C \left(\|f\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}+\alpha})} \|g\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}-\alpha})} \right. \\ &\quad \left. + \|g\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}+\alpha})} \|f\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}-\alpha})} \right) \end{aligned}$$

for all $f, g \in \tilde{L}^{\frac{2}{1\pm\alpha}}(0, T; F\dot{B}_{p,r}^{3-\frac{3}{p}\pm\alpha}(\mathbb{R}^3))$.

Proof. Bony's decomposition (see [5, 8]) for $\Delta_j(fg)$ reads

$$\begin{aligned} \Delta_j(fg) &= \sum_{|k-j|\leq 4} \Delta_j S_{k-1} f \Delta_k g + \sum_{|k-j|\leq 4} \Delta_j S_{k-1} f \Delta_k g + \sum_{k\geq j-2} \sum_{|k'-k|\leq 1} \Delta_j \Delta_k f \Delta_{k'} g \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{2.9}$$

Then by the triangle inequalities in $l^p(\mathbb{Z})$ and $L^p(\mathbb{R})$, we have

$$\begin{aligned} \|fg\|_{\tilde{L}^1(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}})} &\leq \left(\sum_{j\in\mathbb{Z}} 2^{j(3-\frac{3}{p})r} \|\widehat{I}_1\|_{L^1(0,T;L^p)}^r \right)^{\frac{1}{r}} + \left(\sum_{j\in\mathbb{Z}} 2^{j(3-\frac{3}{p})r} \|\widehat{I}_2\|_{L^1(0,T;L^p)}^r \right)^{\frac{1}{r}} \\ &\quad + \left(\sum_{j\in\mathbb{Z}} 2^{j(3-\frac{3}{p})r} \|\widehat{I}_3\|_{L^1(0,T;L^p)}^r \right)^{\frac{1}{r}} \\ &:= J_1 + J_2 + J_3. \end{aligned}$$

For J_1 , Young's inequality and Hölder's inequality ensures that

$$\begin{aligned} 2^{j(3-\frac{3}{p})} \|\widehat{I}_1\|_{L^1(0,T;L^p)} &\leq 2^{j(3-\frac{3}{p})} \left\| \sum_{|k-j|\leq 4} \|\hat{\psi}_j[(\sum_{k'\leq k-2} \hat{\psi}_{k'} \hat{f}) * (\hat{\psi}_k \hat{g})]\|_{L^p} \right\|_{L^1(0,T)} \\ &\leq C 2^{j(3-\frac{3}{p})} \sum_{|k-j|\leq 4} \left(\sum_{k'\leq k-2} 2^{k'(3-\frac{3}{p})} \|\hat{\psi}_{k'} \hat{f}\|_{L^{\frac{2}{1-\alpha}}(0,T;L^p)} \right) \|\hat{\psi}_k \hat{g}\|_{L^{\frac{2}{1+\alpha}}(0,T;L^p)} \\ &\leq C 2^{j(3-\frac{3}{p})} \sum_{|k-j|\leq 4} \left(\sum_{k'\leq k-2} 2^{k'(3-\frac{3}{p}-\alpha)r} \|\hat{\psi}_{k'} \hat{f}\|_{L^{\frac{2}{1-\alpha}}(0,T;L^p)}^r \right)^{\frac{1}{r}} \\ &\quad \times \left(\sum_{k'\leq k-2} 2^{k'\alpha r'} \right)^{\frac{1}{r'}} \|\hat{\psi}_k \hat{g}\|_{L^{\frac{2}{1+\alpha}}(0,T;L^p)} \\ &\leq C \sum_{|k-j|\leq 4} 2^{(j-k)(3-\frac{3}{p})} 2^{k(3-\frac{3}{p}+\alpha)} \|\hat{\psi}_k \hat{g}\|_{L^{\frac{2}{1+\alpha}}(0,T;L^p)} \|f\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}-\alpha})} \end{aligned}$$

since $\alpha > 0$, where $\frac{1}{r} + \frac{1}{r'} = 1$. Hence, by the Young's inequality, we get

$$J_1 \leq C \|f\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}-\alpha})} \|g\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}+\alpha})}.$$

Similarly, we have

$$J_2 \leq C \|f\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}+\alpha})} \|g\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}-\alpha})}.$$

For J_3 , Young's inequality together with Hölder's inequality gives

$$\begin{aligned} 2^{j(3-\frac{3}{p})} \|\widehat{f}_3\|_{L^1(0,T;L^p)} &\leq 2^{j(3-\frac{3}{p})} \left\| \sum_{k \geq j-2} \|\hat{\psi}_j[(\hat{\psi}_k \hat{f}) * (\sum_{|k'-k| \leq 1} \hat{\psi}_{k'} \hat{g})]\|_{L^p} \right\|_{L^1(0,T)} \\ &\leq C 2^{j(3-\frac{3}{p})} \sum_{k \geq j-2} 2^{3k(1-\frac{1}{p})} \|\hat{\psi}_k \hat{f}\|_{L^{\frac{2}{1+\alpha}}(0,T;L^p)} \left(\sum_{|k'-k| \leq 1} 2^{-k'(3-\frac{3}{p}-\alpha)r'} \right)^{\frac{1}{r'}} \\ &\quad \times \left(\sum_{|k'-k| \leq 1} 2^{k'(3-\frac{3}{p}-\alpha)r} \|\hat{\psi}_{k'} \hat{g}\|_{L^{\frac{2}{1-\alpha}}(0,T;L^p)}^r \right)^{\frac{1}{r}} \\ &\leq C \sum_{k \geq j-2} 2^{(j-k)(3-\frac{3}{p})} 2^{k(3-\frac{3}{p}+\alpha)} \|\hat{\psi}_k \hat{f}\|_{L^{\frac{2}{1+\alpha}}(0,T;L^p)} \|g\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}-\alpha})}, \end{aligned}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. Hence, by the Young's inequality, one has

$$\begin{aligned} J_3 &\leq C \left(\sum_{2 \geq k} 2^{k(3-\frac{3}{p})} \right) \|f\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}-\alpha})} \|g\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}+\alpha})} \\ &\leq C \|f\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}-\alpha})} \|g\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}+\alpha})} \end{aligned}$$

since $p > 1$. Summing up, we arrive at

$$\begin{aligned} \|fg\|_{\tilde{L}^1(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}})} &\leq C \left(\|f\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}+\alpha})} \|g\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}-\alpha})} \right. \\ &\quad \left. + \|g\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}+\alpha})} \|f\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{p,r}^{3-\frac{3}{p}-\alpha})} \right) \end{aligned}$$

□

The above lemma excludes the end point case $p = 1$, which is considered in the following one.

Lemma 2.4 Let $T > 0$, $\alpha \in (0, 1]$ and $r \in [1, 2]$. There exists a constant $C > 0$ such that

$$\begin{aligned} \|fg\|_{\tilde{L}^1(0,T;F\dot{B}_{1,r}^0)} &\leq C \left(\|f\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;F\dot{B}_{1,r}^\alpha)} \|g\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{1,r}^{-\alpha})} \right. \\ &\quad \left. + \|g\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;F\dot{B}_{1,r}^\alpha)} \|f\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;F\dot{B}_{1,r}^{-\alpha})} \right) \end{aligned}$$

for all $f, g \in \tilde{L}^{\frac{2}{1\pm\alpha}}(0, T; F\dot{B}_{1,r}^{\pm\alpha}(\mathbb{R}^3))$.

Proof Similar to the proof of Lemma 2.3, by means of Bony's decomposition we have

$$\begin{aligned} \|fg\|_{\tilde{L}^1(0,T;FB_{1,r}^0)} &\leq \left(\sum_{j \in \mathbb{Z}} \|\widehat{I}_1\|_{L^1(0,T;L^1)}^r \right)^{\frac{1}{r}} + \left(\sum_{j \in \mathbb{Z}} \|\widehat{I}_2\|_{L^1(0,T;L^1)}^r \right)^{\frac{1}{r}} \\ &\quad + \left(\sum_{j \in \mathbb{Z}} \|\widehat{I}_3\|_{L^1(0,T;L^1)}^r \right)^{\frac{1}{r}} \\ &:= J_1 + J_2 + J_3, \end{aligned}$$

where $I_i (i = 1, 2, 3)$ are defined in (2.9). It is easy to check that the estimates for J_1 and J_2 in the proof of Lemma 2.3 also hold for the case that $p = 1$ with $\alpha \in (0, 1]$. Thus, we only need to consider J_3 .

Applying Hölder's inequality and Young's inequality gives

$$\begin{aligned} J_3 &\leq \left\| \sum_{k \in \mathbb{Z}} \|[(\hat{\psi}_k \hat{f}) * (\sum_{|k'-k| \leq 1} \hat{\psi}_{k'} \hat{g})]\|_{L^1} \right\|_{L^1(0,T)} \\ &\leq \left(\sum_{k \in \mathbb{Z}} 2^{-k\alpha r} \|\hat{\psi}_k \hat{f}\|_{L^{\frac{2}{1-\alpha}}(0,T;L^1)}^r \right)^{\frac{1}{r}} \left\| \left(\sum_{|k'-k| \leq 1} 2^{(k-k')\alpha} 2^{k'\alpha} \|\hat{\psi}_{k'} \hat{g}\|_{L^{\frac{2}{1+\alpha}}(0,T;L^1)} \right) \right\|_{l^{r'}} \\ &\leq \|f\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;FB_{1,r}^{-\alpha})} \left(\sum_{|k-k'| \leq 1} 2^{k'\alpha r} \|\hat{\psi}_{k'} \hat{g}\|_{L^{\frac{2}{1+\alpha}}(0,T;L^1)}^r \right)^{\frac{1}{r}} \left(\sum_{|k| \leq 1} 2^{k\alpha m} \right)^{\frac{1}{m}} \\ &\leq C \|f\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;FB_{1,r}^{-\alpha})} \|g\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;FB_{1,r}^{\alpha})}, \end{aligned}$$

where the first inequality has used $l^1 \hookrightarrow l^r$ and $\sum_{j \in \mathbb{Z}} \hat{\psi}_j = 1$, $\frac{1}{r} + \frac{1}{r'} = 1$ and $m \in [1, \infty]$ satisfying $\frac{1}{r} + \frac{1}{m} = 1 + \frac{1}{r'}$ for $r \in [1, 2]$.

2.3 Proofs of Theorems 1.4 and 1.5

The proofs of Theorems 1.4 and 1.5 follow from the following standard Banach fixed point lemma combined with the estimates established in the previous section.

Lemma 2.5 (Cannone and Karch [10]) Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space and $B : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ a bounded bilinear form satisfying $\|B(x_1, x_2)\|_{\mathcal{X}} \leq \eta \|x_1\|_{\mathcal{X}} \|x_2\|_{\mathcal{X}}$ for all $x_1, x_2 \in \mathcal{X}$ and some constant $\eta > 0$. Then, if $0 < \varepsilon < \frac{1}{4\eta}$ and if $y \in \mathcal{X}$ such that $\|y\|_{\mathcal{X}} < \varepsilon$, the equation $x = y + B(x, x)$ has a solution in \mathcal{X} such that $\|x\|_{\mathcal{X}} \leq 2\varepsilon$. This solution is the only one in the ball $\bar{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the following sense: if $\|\tilde{y}\|_{\mathcal{X}} \leq \varepsilon$, $\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x})$ and $\|\tilde{x}\|_{\mathcal{X}} \leq 2\varepsilon$ then

$$\|x - \tilde{x}\|_{\mathcal{X}} \leq \frac{1}{1 - 4\eta\varepsilon} \|y - \tilde{y}\|_{\mathcal{X}}.$$

□

Proof of Theorem 1.4 Define

$$B(v, w)(t) := \int_0^t T_{\Omega, N}(t - \tau) \tilde{\mathbb{P}} \tilde{\nabla} \cdot [v(\tau) \otimes w(\tau)] d\tau. \quad (2.10)$$

Let $\alpha \in (0, 1)$ be any given and fixed. For $T > 0$ to be specified later, let X_T^α be the function space introduced in Theorem 1.4. It is clear that X_T^α is a Banach space endowed with the norm

$$\|v\|_{X_T^\alpha} := \|v\|_{\tilde{L}^{\frac{2}{1-\alpha}}(0,T;FB_{p,r}^{3-\frac{3}{p}-\alpha})} + \|v\|_{\tilde{L}^{\frac{2}{1+\alpha}}(0,T;FB_{p,r}^{3-\frac{3}{p}+\alpha})}.$$

By applying Lemmas 2.2 and 2.3 with $s = 2 - \frac{3}{p}$ and $\rho = 1$, we arrive at

$$\begin{aligned} \|B(v, w)\|_{X_T^\alpha} &= \left\| \int_0^t T_{\Omega, N}(t - \tau) \tilde{\mathbb{P}} \tilde{\nabla} \cdot [v(\tau) \otimes w(\tau)] d\tau \right\|_{X_T^\alpha} \\ &\leq C_0 (\nu^{-\frac{1-\alpha}{2}} + \nu^{-\frac{1+\alpha}{2}}) \|\tilde{\nabla} \cdot [v(\tau) \otimes w(\tau)]\|_{\tilde{L}^1(0,T;FB_{p,r}^{2-\frac{3}{p}})} \\ &\leq C_0 \max\{\nu^{-\frac{1+\alpha}{2}}, \nu^{-\frac{1-\alpha}{2}}\} \|v\|_{X_T^\alpha} \|w\|_{X_T^\alpha} \end{aligned}$$

for $v, w \in X_T^\alpha$ and some constant $C_0 > 0$. Now let $v_0 \in FB_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3)$ be given, then there exists $T > 0$ such that

$$\|T_{\Omega, N}(t)v_0\|_{X_T^\alpha} \leq \frac{\min\{\nu^{\frac{1+\alpha}{2}}, \nu^{\frac{1-\alpha}{2}}\}}{8C_0}. \quad (2.11)$$

With $T > 0$ specified in this way, Lemma 2.5 ensures that there exists a unique solution v of (2.9) in the ball with center 0 and radius $\frac{\min\{\nu^{\frac{1+\alpha}{2}}, \nu^{\frac{1-\alpha}{2}}\}}{2C_0}$ in the space X_T^α . Moreover, applying Lemmas 2.1 \sim 2.3 with $s = 2 - \frac{3}{p}$ and $\rho = 1$ leads to

$$\begin{aligned} \|v\|_{\tilde{L}^\infty(0,T;FB_{p,r}^{2-\frac{3}{p}})} &\leq C \|v_0\|_{FB_{p,r}^{2-\frac{3}{p}}} + C \|\tilde{\nabla} \cdot [v(\tau) \otimes v(\tau)]\|_{\tilde{L}^1(0,T;FB_{p,r}^{2-\frac{3}{p}})} \\ &\leq C \|v_0\|_{FB_{p,r}^{2-\frac{3}{p}}} + C \|v\|_{X_T^\alpha}^2 < \infty. \end{aligned}$$

By using a standard density argument, we can further infer that $v \in [C([0, T], FB_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3))]^4$. This proves the local well-posedness assertion in Theorem 1.4.

Next we assume that the condition (2.11) is satisfied. It follows from Lemma 2.1 with $s = 2 - \frac{3}{p}$ that there exists a constant $C_1 > 0$ such that

$$\|T_{\Omega, N}(t)v_0\|_{X_T^\alpha} \leq C_1 \max\{\nu^{-\frac{(1+\alpha)}{2}}, \nu^{-\frac{(1-\alpha)}{2}}\} \|v_0\|_{FB_{p,r}^{2-\frac{3}{p}}}$$

for any given $T > 0$. Hence, if

$$\|u_0\|_{FB_{p,r}^{2-\frac{3}{p}}} \leq \frac{\nu}{4C_0C_1},$$

then the smallness condition (2.11) with X_T^α replaced by X_∞^α is satisfied. Then by deducing similarly as above we see that problem (2.9) has a unique solution $v \in [C([0, \infty); FB_{p,r}^{2-\frac{3}{p}}(\mathbb{R}^3))]^4 \cap X_\infty^\alpha$. This proves the global well-posedness assertion in Theorem 1.4 and finishes the proof of Theorem 1.4. \square

Proof of Theorem 1.5: The proof of Theorem 1.5 is quite similar to that of Theorem 1.4. What we only need to modify is to replace the function spaces X_T^α and X_∞^α respectively with Y_T^α and Y_∞^α introduced in Theorem 1.5. We omit the details.

3 Proof of Theorem 1.6

In this section we give the proof of Theorem 1.6. Similar to [35], we shall use the following abstract result of Bejenaru and Tao [6] to prove Theorem 1.6.

Consider the abstract equation

$$u = L(f) + N_k(u, \dots, u), \quad (3.1)$$

where the initial data f takes values in some data space D , the solution u takes values in some solution space S , the linear operator $L : \mathcal{D}(L) \subseteq D \rightarrow S$ is densely defined, and the k -linear operator $N_k : \mathcal{D}(N_k) \subseteq S \times \dots \times S \rightarrow S$ with $k \geq 2$ is also densely defined. And let $(A_{n_1}(f), \dots, A_{n_k}(f)) \in \mathcal{D}(N_k)$ for all $f \in \mathcal{D}(L)$, $k \geq 2$ and $n_1, n_2, \dots, n_k \in \mathbb{N}$, where

$$\begin{cases} A_1(f) := L(f) \\ A_n(f) := \sum_{n_1 + \dots + n_k = n} N_k(A_{n_1}(f), \dots, A_{n_k}(f)) \quad \text{for } n = 2, 3, \dots \end{cases}$$

Proposition 3.1 (Bejenaru and Tao [6]) Suppose that Eq. (3.1) is quantitatively well-posed in the Banach spaces D and S in the sense that there exists a solution map $f \mapsto u[f]$ from a ball B_D in D to a ball B_S in S which is continuous with respect to the norm topologies of D and S . Suppose that these spaces are endowed with different norms to form different normed vector spaces D' and S' (not necessarily complete), respectively, which are weaker than D and S in the sense that there exists a constant $C > 0$ such that

$$\|f\|_{D'} \leq C\|f\|_D \quad (\text{for } f \in D) \quad \text{and} \quad \|u\|_{S'} \leq C\|u\|_S \quad (\text{for } u \in S).$$

Suppose further that the solution map $f \mapsto u[f]$ is continuous from $(B_D, \|\cdot\|_{D'})$ (i.e. the ball B_D equipped with the D' topology) to $(B_S, \|\cdot\|_{S'})$. Then for each n , the non-linear operator A_n is continuous from $(B_D, \|\cdot\|_{D'})$ to $(S, \|\cdot\|_{S'})$.

Proof of Theorem 1.6: We shall use Proposition 3.1 with the help of Theorem 1.5 to prove Theorem 1.6. Define $D := [F\dot{B}_{1,2}^{-1}(\mathbb{R}^3)]^4$ and $S := [C([0, \infty); F\dot{B}_{1,2}^{-1}(\mathbb{R}^3))]^4 \cap [\tilde{L}^{\frac{2}{1+\alpha}}(0, \infty; F\dot{B}_{1,2}^\alpha(\mathbb{R}^3))]^4 \cap [\tilde{L}^{\frac{2}{1-\alpha}}(0, \infty; F\dot{B}_{1,2}^{-\alpha}(\mathbb{R}^3))]^4$. Theorem 1.5 implies that that for any $0 < \alpha < 1$ there exists corresponding $\varepsilon > 0$ such that there exists a solution map $B_D \ni f \mapsto v[f] \in S$ which is continuous with respect to the norm topologies of D and S , where

$$B_D := \{f \in D \mid \|f\|_D \leq \varepsilon\} \quad \text{and} \quad v[f] := T_{\Omega, N}f - B(v[f], v[f]),$$

where $B(v, w)$ is defined in (2.10). Let $D' := [F\dot{B}_{1,r}^{-1}(\mathbb{R}^3)]^4$ and $S' := [L^\infty(0, \infty; F\dot{B}_{1,r}^{-1})]^4$ with $2 < r \leq \infty$. It is obvious that the embeddings $D \hookrightarrow D'$ and $S \hookrightarrow S'$ are continuous.

We shall prove by contradiction that the solution map $(B_D, \|\cdot\|_{D'}) \ni f \mapsto v[f] \in (S, \|\cdot\|_{S'})$ is not continuous, no matter how small ε is chosen. Hence if we assume that $(B_D, \|\cdot\|_{D'}) \ni$

$f \mapsto v[f] \in (S, \|\cdot\|_{S'})$ is continuous for some $\varepsilon > 0$. Then, by Proposition 3.1, the map $(B_D, \|\cdot\|_{D'}) \ni f \mapsto A_2(f) \in (S, \|\cdot\|_{S'})$ is also continuous, where

$$A_2(f) := B(T_{\Omega,N}(\cdot)f, T_{\Omega,N}(\cdot)f).$$

However, in what follows we shall construct a sequence $\{f^M\}_{M=1}^\infty \in B_D$ such that

$$\|f^M\|_{D'} \rightarrow 0 \quad \text{as } M \rightarrow \infty, \quad (3.2)$$

and there exists a constant $c > 0$ independent of $M \in \mathbb{N}$ such that

$$\|A_2(f^M)\|_{S'} = \|B(T_{\Omega,N}(\cdot)f^M, T_{\Omega,N}(\cdot)f^M)\|_{S'} \geq c \quad (3.3)$$

for all sufficiently large M , and a contradiction follows.

We now construct our counterexample $\{f^M\}_{M=1}^\infty$. Define

$$\chi(\xi) = \begin{cases} 1, & \text{if } |\xi_k| \leq 1, k = 1, 2, 3, \\ 0, & \text{otherwise,} \end{cases}$$

and $\chi_j^\pm(\xi) = \chi(\xi \mp 2^j e_2)$ for $j \in \mathbb{Z}$, where $e_2 = (0, 1, 0)$. Then, we set the initial data $\{f^M\}_{M=1}^\infty$ via Fourier transform as follows:

$$\widehat{f^M}(\xi) := \frac{i}{M^{\frac{1}{2}}} \sum_{j=M}^{2M} 2^j (\chi_j^+(\xi) + \chi_j^-(\xi)) \frac{1}{|\xi|} \begin{pmatrix} \xi_2 \\ -\xi_1 \\ 0 \\ 0 \end{pmatrix}.$$

By the definition of norm of $F B_{1,r}^{-1}$, there exists $C > 0$ such that

$$\|f^M\|_{F B_{1,r}^{-1}} \leq C M^{-\frac{1}{2} + \frac{1}{r}}, \quad \text{for } M \in \mathbb{N} \text{ and } 2 \leq r \leq \infty.$$

Thus $\varepsilon C^{-1} f^M \in B_D$ and (3.2) is satisfied for all $2 < r \leq \infty$. Hereinafter, we omit the absolute constant εC^{-1} for simplicity and prove the property (3.3).

Let E be a measurable set in \mathbb{R}^3 such that the Lebesgue measure of E is positive, there exists a constant $c > 0$ such that

$$1 - \frac{\xi_1^2}{|\xi|^2} \geq c, \quad \text{for } \xi = (\xi_1, \xi_2, \xi_3) \in E, \quad (3.4)$$

and

$$E \subset \left\{ \xi \in \mathbb{R}^3 \mid \frac{1}{1000} \leq \xi_1 \leq 1, |\xi| \leq 1 \right\}. \quad (3.5)$$

Since E is bounded, there exists $j_0 \in \mathbb{N}$ such that $\sum_{j=-j_0}^{j_0} \hat{\psi}_j(\xi) = 1$ for $\xi \in E$. Thus there exists a constant $C_E > 0$ such that

$$\|A_2(f^M)(t)\|_{F B_{1,r}^{-1}} \geq C_E \|\mathcal{F}[A_2(f^M)](t)\|_{L^1(E)}, \quad \text{for all } t \geq 0. \quad (3.6)$$

In what follows, we just need to estimate $\|\mathcal{F}[A_2(f^M)](t)\|_{L^1(E)}$. Now, we assume that $N > 0$. If $N = 0$, combining with $\theta_0 = (f^M)_4 = 0$, we have $\theta \equiv 0$. Then it becomes Cauchy problem of (1.3) which have discussed in [35]. Furthermore, we assume that $N \geq |\Omega| \geq 0$. The argument of the case $|\Omega| \geq N > 0$ is similar.

For the sake of convenience in writing, we define \hat{T}_i , $i = 1, 2, 3$ by

$$\hat{T}_1(\xi, t) := \cos\left(\frac{|\xi|'}{|\xi|}t\right)e^{-\nu|\xi|^2t}, \quad \hat{T}_2(\xi, t) := \sin\left(\frac{|\xi|'}{|\xi|}t\right)e^{-\nu|\xi|^2t}, \quad (3.7)$$

and

$$\hat{T}_3(\xi, t) := e^{-\nu|\xi|^2t}, \quad (3.8)$$

where $|\xi|$ and $|\xi|'$ are defined as in (2.4). Moreover, we have the following key observations:

- $\tilde{\mathbb{P}} : [\mathcal{S}'(\mathbb{R}^3)]^4 \mapsto \mathcal{S}'_\sigma(\mathbb{R}^3) \times \mathcal{S}'(\mathbb{R}^3)$;
- $(M_1 + M_3)(\hat{v}) = \hat{v}$ for $v \in \mathcal{S}'_\sigma(\mathbb{R}^3) \times \mathcal{S}'(\mathbb{R}^3)$, since

$$M_1 + M_3 = \begin{pmatrix} 1 - \frac{N^2\xi_1^2}{|\xi|'^2} & -\frac{N^2\xi_1\xi_2}{|\xi|'^2} & -\frac{N^2\xi_1\xi_3}{|\xi|'^2} & 0 \\ -\frac{N^2\xi_1\xi_2}{|\xi|'^2} & 1 - \frac{N^2\xi_2^2}{|\xi|'^2} & -\frac{N^2\xi_2\xi_3}{|\xi|'^2} & 0 \\ -\frac{N^2\xi_1\xi_3}{|\xi|'^2} & -\frac{N^2\xi_2\xi_3}{|\xi|'^2} & 1 - \frac{N^2\xi_3^2}{|\xi|'^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\mathcal{S}'_\sigma(\mathbb{R}^3) := \{u \in [\mathcal{S}'(\mathbb{R}^3)]^3 : \operatorname{div} u = 0\}$, M_1 , M_3 and $\tilde{\mathbb{P}}$ are defined as (2.5) (2.7) and (2.8), respectively.

By the similar argument of [35], we arrive at

$$\begin{aligned} & |\mathcal{F}[A_2(f^M)](t)(\xi)| \\ & \geq \left| \int_0^t \hat{T}_1(\xi, t - \tau) \sum_{l=1}^3 \left(\delta_{1,l} - \frac{\xi_1\xi_l}{|\xi|^2} \right) \sum_{k=1}^3 \xi_k [(T_{\Omega,N}(\tau)f^M)_k * (T_{\Omega,N}(\tau)f^M)_l] d\tau \right| \\ & \quad - \left| \int_0^t \hat{T}_2(\xi, t - \tau) M_2 \mathcal{F}[\tilde{\mathbb{P}}\tilde{\nabla} \cdot (T_{\Omega,N}(\tau)f^M \otimes T_{\Omega,N}(\tau)f^M)](\xi) d\tau \right| \\ & \quad - \left| \int_0^t [\hat{T}_3(\xi, t - \tau) - \hat{T}_1(\xi, t - \tau)] M_3 \mathcal{F}[\tilde{\mathbb{P}}\tilde{\nabla} \cdot (T_{\Omega,N}(\tau)f^M \otimes T_{\Omega,N}(\tau)f^M)](\xi) d\tau \right| \\ & =: K_1(\xi, t) - K_2(\xi, t) - K_3(\xi, t). \end{aligned} \quad (3.9)$$

Here we have used the fact that $(M_1 + M_3)\mathcal{F}[\tilde{\mathbb{P}}f] = \mathcal{F}[\tilde{\mathbb{P}}f]$ for $f \in [\mathcal{S}'(\mathbb{R}^3)]^4$. In the following, we divide our calculations into two parts:

(i) Estimates for $K_1(\xi, t)$ with $\xi \in E$

By the definition of $T_{\Omega,N}(\cdot)$, we see

$$\begin{aligned}
K_1(\xi, t) &\geq \left| \int_0^t \hat{T}_1(\xi, t-\tau) \left(1 - \frac{\xi_1^2}{|\xi|^2}\right) \xi_1 [(T_{\Omega,N}(\tau)f^M)_1 * (T_{\Omega,N}(\tau)f^M)_1] d\tau \right| \\
&\quad - \sum_{k=2,3} \left| \int_0^t \hat{T}_1(\xi, t-\tau) \left(1 - \frac{\xi_1^2}{|\xi|^2}\right) \xi_k [(T_{\Omega,N}(\tau)f^M)_k * (T_{\Omega,N}(\tau)f^M)_1] d\tau \right| \\
&\quad - \sum_{k=1,2,3, l=2,3} \left| \int_0^t \hat{T}_1(\xi, t-\tau) \frac{\xi_1 \xi_l}{|\xi|^2} \xi_k [(T_{\Omega,N}(\tau)f^M)_k * (T_{\Omega,N}(\tau)f^M)_l] d\tau \right| \\
&=: K_{111}(\xi, t) - \sum_{k=2,3} K_{1k1}(\xi, t) - \sum_{k=1,2,3, l=2,3} K_{1kl}(\xi, t). \tag{3.10}
\end{aligned}$$

On the estimate of $K_{111}(\xi, t)$ for $\xi \in E$. By the definition of $T_{\Omega,N}$, one has

$$\begin{aligned}
K_{111}(\xi, t) &\geq \left| \int_0^t \hat{T}_1(\xi, t-\tau) \left(1 - \frac{\xi_1^2}{|\xi|^2}\right) \xi_1 [(\hat{T}_3(\xi, \tau)M_3\widehat{f^M})_1 * (\hat{T}_3(\xi, \tau)M_3\widehat{f^M})_1] d\tau \right| \\
&\quad - \sum_{k=1,2} \left| \int_0^t \hat{T}_1(\xi, t-\tau) \left(1 - \frac{\xi_1^2}{|\xi|^2}\right) \xi_1 [(\hat{T}_k(\xi, \tau)M_k\widehat{f^M})_1 * (\hat{T}_3(\xi, \tau)M_3\widehat{f^M})_1] d\tau \right| \\
&\quad - \sum_{k=1,2,3, l=1,2} \left| \int_0^t \hat{T}_1(\xi, t-\tau) \left(1 - \frac{\xi_1^2}{|\xi|^2}\right) \xi_1 [(\hat{T}_k(\xi, \tau)M_k\widehat{f^M})_1 * (\hat{T}_l(\xi, \tau)M_l\widehat{f^M})_1] d\tau \right| \\
&=: J_{133} - \sum_{k=1,2} J_{1k3} - \sum_{k=1,2,3, l=1,2} J_{1kl}. \tag{3.11}
\end{aligned}$$

We now estimate each term. It follows from the definitions of f^M that

$$\begin{aligned}
J_{133} &= \left| 2 \int_0^t \hat{T}_1(\xi, t-\tau) \left(1 - \frac{\xi_1^2}{|\xi|^2}\right) \xi_1 \int_{\mathbb{R}^3} e^{-\nu|\xi-\eta|^2\tau} e^{-\nu|\eta|^2\tau} \frac{N^2|\xi_h - \eta_h|^2}{|\xi - \eta|'^2} \frac{\xi_2 - \eta_2}{|\xi - \eta|} \right. \\
&\quad \left. \times \frac{N^2|\eta_h|^2}{|\eta|'^2} \frac{\eta_2}{|\eta|} \frac{1}{M} \sum_{j=M}^{2M} 2^{2j} \chi_j^+(\xi - \eta) \chi_j^-(\eta) d\eta d\tau \right|,
\end{aligned}$$

where we have used the support properties of χ_j^\pm and $\eta_h := (\eta_1, \eta_2)$ for every $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$.

Since

$$\frac{|\xi|'}{|\xi|}(t-\tau) \leq Nt \leq 1 \quad \text{and} \quad e^{-\frac{\nu}{N}} \leq e^{-\nu|\xi|^2(t-\tau)}$$

for $t \in (0, \frac{1}{N}]$ and $\xi \in E$, there exists a constant $0 < c < 1$ such that

$$\hat{T}_1(\xi, t-\tau) \geq c,$$

and since

$$-\frac{N^4}{\Omega^4} \leq \frac{N^2|\xi_h - \eta_h|^2}{|\xi - \eta|'^2} \frac{\xi_2 - \eta_2}{|\xi - \eta|} \frac{N^2|\eta_h|^2}{|\eta|'^2} \frac{\eta_2}{|\eta|} \leq -\frac{1}{256}$$

for $\eta \in \text{supp}\chi_j^-$ with $\xi - \eta \in \text{supp}\chi_j^+$, or $\eta \in \text{supp}\chi_j^+$ with $\xi - \eta \in \text{supp}\chi_j^-$, we then obtain

$$\begin{aligned} J_{133} &\geq \frac{c}{M} \sum_{j=M}^{2M} 2^{2j} \int_0^t \int_{\mathbb{R}^3} e^{-\nu|\xi-\eta|^2\tau} e^{-\nu|\eta|^2\tau} \chi_j^+(\xi - \eta) \chi_j^-(\eta) d\eta d\tau \\ &\geq \frac{c}{M} \sum_{j=M}^{2M} 2^{2j} 2^{-2j} (1 - e^{-\nu 4t 2^{2j}}) \geq c \end{aligned}$$

for $\xi \in E$ and $\nu^{-1} 2^{-2M} \leq t \leq \frac{1}{N}$ with $M \geq \frac{1}{2} \log_2 \frac{N}{\nu}$, which yields that

$$\|J_{133}\|_{L^1(E)} \geq c. \quad (3.12)$$

For J_{113} , we have for $\xi \in E$ that

$$\begin{aligned} J_{113} &\leq C \left| \int_0^t \int_{\mathbb{R}^3} e^{-\nu(|\xi-\eta|^2+|\eta|^2)\tau} \frac{\Omega^2|\xi_3-\eta_3|^2}{|\xi-\eta|'^2} \frac{N^2|\eta_h|^2}{|\eta|'^2} \frac{\xi_2-\eta_2}{|\xi-\eta|} \frac{\eta_2}{|\eta|} \right. \\ &\quad \left. \times \frac{1}{M} \sum_{j=M}^{2M} 2^{2j} \chi_j^+(\xi - \eta) \chi_j^-(\eta) d\eta d\tau \right| \\ &\leq \frac{C}{M} \frac{N^2}{\Omega^2} \sum_{j=M}^{2M} 2^{2j} 2^{-2j} \int_{\mathbb{R}^3} \frac{1 - e^{-\nu(|\xi-\eta|^2+|\eta|^2)t}}{\nu(|\xi-\eta|^2+|\eta|^2)} \chi_j^+(\xi - \eta) \chi_j^-(\eta) d\eta d\tau \\ &\leq \frac{C}{M} \frac{N^2}{\Omega^2} \sum_{j=M}^{2M} 2^{2j} 2^{-2j} 2^{-2j} \leq \frac{CN^2}{\Omega^2 M 2^{2M}}. \end{aligned} \quad (3.13)$$

Similarly,

$$\begin{aligned} J_{111} &\leq C \left| \int_0^t \int_{\mathbb{R}^3} e^{-\nu(|\xi-\eta|^2+|\eta|^2)\tau} \times \frac{\Omega^2|\xi_3-\eta_3|^2}{|\xi-\eta|'^2} \frac{\xi_2-\eta_2}{|\xi-\eta|} \frac{\Omega^2\eta_3^2}{|\eta|'^2} \frac{\eta_2}{|\eta|} \right. \\ &\quad \left. \times \frac{1}{M} \sum_{j=M}^{2M} 2^{2j} \chi_j^+(\xi - \eta) \chi_j^-(\eta) d\eta d\tau \right| \\ &\leq \frac{C}{M 2^{4M}}. \end{aligned} \quad (3.14)$$

Thus, noting that $J_{131} = J_{113}$, it follows from (3.13) and (3.14) that

$$\|J_{111}\|_{L^1(E)} + \|J_{113}\|_{L^1(E)} + \|J_{131}\|_{L^1(E)} \leq \frac{CN^2}{\Omega^2 M 2^{2M}}. \quad (3.15)$$

For J_{112} , we note that

$$|\hat{T}_2(\zeta, \tau)| \leq \frac{|\zeta|'}{|\zeta|} \tau \leq Nt \quad (3.16)$$

for all $\tau \in (0, t)$ and all $\zeta \in \mathbb{R}^3$. Hence, for $\xi \in E$ one has

$$J_{112} \leq Ct \int_0^t \int_{\mathbb{R}^3} e^{-\nu(|\xi-\eta|^2+|\eta|^2)\tau} |\widehat{f^M}(\xi - \eta)| |\widehat{f^M}(\eta)| d\eta d\tau. \quad (3.17)$$

Since it holds that $\sum_{j=-j_0}^{j_0} \hat{\psi}_j(\xi) = 1$ for all $\xi \in E$, Young's inequality together with Hölder inequality, Lemma 2.1 and Lemma 2.4 ensures that

$$\begin{aligned}
\|J_{112}\|_{L^1(E)} &\leq Ct \left\| \int_0^t \left[e^{-\nu|\cdot|^2\tau} |\widehat{f^M}(\cdot)| \right] * \left[e^{-\nu|\cdot|^2\tau} |\widehat{f^M}(\cdot)| \right] d\tau \right\|_{L^1} \\
&\leq Ct \left\{ \sum_{j=-j_0}^{j_0} \left(\int_0^t \left\| \hat{\psi}_j \mathcal{F} \left[\mathcal{F}^{-1} [e^{-\nu|\cdot|^2\tau} |\widehat{f^M}|]^2 \right] \right\|_{L^1} d\tau \right)^2 \right\}^{\frac{1}{2}} \\
&\leq Ct \left\| \mathcal{F}^{-1} [e^{-\nu|\cdot|^2\tau} |\widehat{f^M}|] \right\|_{L^{\frac{1}{1-\alpha}}(0,\infty; FB_{1,2}^{-\alpha})} \left\| \mathcal{F}^{-1} [e^{-\nu|\cdot|^2\tau} |\widehat{f^M}|] \right\|_{L^{\frac{1}{1+\alpha}}(0,\infty; FB_{1,2}^{\alpha})} \\
&\leq Ct \|f^M\|_{FB_{1,2}^{-1}}^2 \leq Ct.
\end{aligned} \tag{3.18}$$

Similarly, we get

$$\|J_{121}\|_{L^1(E)} + \|J_{123}\|_{L^1(E)} + \|J_{132}\|_{L^1(E)} \leq Ct \tag{3.19}$$

and

$$\begin{aligned}
\|J_{122}\|_{L^1(E)} &\leq Ct^2 \left\| \int_0^t \left[e^{-\nu|\cdot|^2\tau} |\widehat{f^M}(\cdot)| \right] * \left[e^{-\nu|\cdot|^2\tau} |\widehat{f^M}(\cdot)| \right] d\tau \right\|_{L^1} \\
&\leq Ct^2.
\end{aligned} \tag{3.20}$$

Therefore, by (3.12), (3.15) and (3.18)-(3.20), we have

$$\|K_{111}\|_{L^1(E)} \geq c - \frac{CN^2}{\Omega^2 M 2^{2M}} - C(t + t^2). \tag{3.21}$$

On the estimate of $K_{121}(\xi, t)$ and $K_{112}(\xi, t)$ for $\xi \in E$. By the definition of $T_{\Omega, N}(\cdot)$, we see

$$\begin{aligned}
K_{121}(\xi, t) &\leq \sum_{k,l=1,2,3} \left| \int_0^t [(\hat{T}_k(\xi, \tau) M_k \widehat{f^M})_2 * (\hat{T}_l(\xi, \tau) M_l \widehat{f^M})_1] d\tau \right| \\
&=: \sum_{k,l=1,2,3} L_{1kl}.
\end{aligned} \tag{3.22}$$

The similar process for getting (3.13) gives

$$\begin{aligned}
L_{111} &\leq \left| -2 \int_0^t \int_{\mathbb{R}^3} e^{-\nu(|\xi-\eta|^2+|\eta|^2)\tau} \frac{\Omega^2(\xi_3-\eta_3)^2}{|\xi-\eta|'^2} \frac{\xi_1-\eta_1}{|\xi-\eta|} \frac{\Omega^2\eta_3^2}{|\eta|'^2} \frac{\eta_2}{|\eta|} \right. \\
&\quad \left. \times \frac{1}{M} \sum_{j=M}^{2M} 2^{2j} \chi_j^+(\xi-\eta) \chi_j^-(\eta) d\eta d\tau \right| \\
&\leq \frac{C}{M 2^{5M}}.
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
L_{113} = L_{131} &\leq \left| 2 \int_0^t \int_{\mathbb{R}^3} e^{-\nu(|\xi-\eta|^2+|\eta|^2)\tau} \frac{N^2|\xi_h-\eta_h|^2}{|\xi-\eta|'^2} \frac{\xi_1-\eta_1}{|\xi-\eta|} \frac{\Omega^2\eta_3^2}{|\eta|'^2} \frac{\eta_2}{|\eta|} \right. \\
&\quad \left. \times \frac{1}{M} \sum_{j=M}^{2M} 2^{2j} \chi_j^+(\xi-\eta) \chi_j^-(\eta) d\eta d\tau \right| \\
&\leq \frac{CN^2}{\Omega^2 M 2^{3M}},
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
L_{133} &\leq \left| 2 \int_0^t \int_{\mathbb{R}^3} e^{-\nu(|\xi-\eta|^2+|\eta|^2)\tau} \frac{N^2|\xi_h-\eta_h|^2}{|\xi-\eta|'^2} \frac{\xi_1-\eta_1}{|\xi-\eta|} \frac{N^2|\eta_h|^2}{|\eta|'^2} \frac{\eta_2}{|\eta|} \right. \\
&\quad \times \left. \frac{1}{M} \sum_{j=M}^{2M} 2^{2j} \chi_j^+(\xi-\eta) \chi_j^-(\eta) d\eta d\tau \right| \\
&\leq \frac{CN^4}{\Omega^4 M 2^M},
\end{aligned} \tag{3.25}$$

which yield that

$$\left\| \sum_{k,l=1,3} L_{1kl} \right\|_{L^1(E)} \leq \frac{CN^4}{\Omega^4 M 2^M}. \tag{3.26}$$

And the similar arguments for getting (3.18) and (3.20) give rise to

$$\begin{aligned}
\left\| \sum_{l=1,2,3} L_{12l} + \sum_{k=1,3} L_{1k2} \right\|_{L^1(E)} &\leq C(t+t^2) \left\| \int_0^t \left[[e^{-\nu|\cdot|^2\tau} \widehat{f^M}(\cdot)] * [e^{-\nu|\cdot|^2\tau} \widehat{f^M}(\cdot)] \right] d\tau \right\|_{L^1} \\
&\leq C(t+t^2).
\end{aligned} \tag{3.27}$$

Therefore, it follows from (3.26) and (3.27) that

$$\|K_{121}\|_{L^1(E)} \leq \frac{CN^4}{\Omega^4 M 2^M} + C(t+t^2). \tag{3.28}$$

Similarly,

$$\|K_{112}\|_{L^1(E)} \leq \frac{CN^4}{\Omega^4 M 2^M} + C(t+t^2). \tag{3.29}$$

On the estimate of $K_{122}(\xi, t)$ for $\xi \in E$. By the definition of $T_{\Omega, N}(\cdot)$, we see

$$\begin{aligned}
K_{122}(\xi, t) &\leq \sum_{k,l=1,2,3} \left| \int_0^t [(\hat{T}_k(\xi, \tau) M_k \widehat{f^M})_2 * (\hat{T}_l(\xi, \tau) M_l \widehat{f^M})_2] d\tau \right| \\
&=: \sum_{k,l=1,2,3} J_{2kl}.
\end{aligned} \tag{3.30}$$

Similar calculations for getting (3.13) lead to

$$\begin{aligned}
\sum_{k,l=1,3} J_{2kl} &\leq \left| -2 \int_0^t \int_{\mathbb{R}^3} e^{-\nu(|\xi-\eta|^2+|\eta|^2)\tau} \left\{ \frac{N^2|\xi_h-\eta_h|^2}{|\xi-\eta|'^2} \frac{N^2|\eta_h|^2}{|\eta|'^2} \right. \right. \\
&\quad + 2 \frac{N^2|\xi_h-\eta_h|^2}{|\xi-\eta|'^2} \frac{\Omega^2 \eta_3^2}{|\eta|'^2} + \frac{\Omega^2(\xi_3-\eta_3)^2}{|\xi-\eta|'^2} \frac{\Omega^2 \eta_3^2}{|\eta|'^2} \left. \right\} \frac{\xi_1-\eta_1}{|\xi-\eta|} \frac{\eta_1}{|\eta|} \\
&\quad \times \frac{1}{M} \sum_{j=M}^{2M} 2^{2j} \chi_j^+(\xi-\eta) \chi_j^-(\eta) d\eta d\tau \Big| \\
&\leq \frac{CN^4}{\Omega^4 M 2^{2M}},
\end{aligned} \tag{3.31}$$

which yields that

$$\left\| \sum_{k,l=1,3} J_{2kl} \right\|_{L^1(E)} \leq \frac{CN^4}{\Omega^4 M 2^{2M}}. \quad (3.32)$$

And similar to (3.27), one has

$$\left\| \sum_{l=1,2,3} J_{22l} + \sum_{k=1,3} J_{2k2} \right\|_{L^1(E)} \leq C(t + t^2). \quad (3.33)$$

Therefore, it follows from (3.32) and (3.33) that

$$\|K_{122}\|_{L^1(E)} \leq \frac{CN^4}{\Omega^4 M 2^{2M}} + C(t + t^2). \quad (3.34)$$

On the estimates of $K_{131}(\xi, t)$, $K_{113}(\xi, t)$, $K_{123}(\xi, t)$, $K_{132}(\xi, t)$ and $K_{133}(\xi, t)$ for $\xi \in E$. By the definition of $T_{\Omega, N}(\cdot)$, together with the fact that $(M_1 f^M)_3 = 0$ and $(M_3 \hat{v})_3 = 0$ for all $v \in [\mathcal{S}'(\mathbb{R}^3)]^4$, one has

$$K_{131}(\xi, t) \leq \sum_{l=1,2,3} \left| \int_0^t [(\hat{T}_2(\xi, \tau) M_2 \widehat{f^M})_3 * (\hat{T}_l(\xi, \tau) M_l \widehat{f^M})_1] d\tau \right|, \quad (3.35)$$

$$K_{123}(\xi, t) \leq \sum_{k=1,2,3} \left| \int_0^t [(\hat{T}_k(\xi, \tau) M_k \widehat{f^M})_2 * (\hat{T}_2(\xi, \tau) M_2 \widehat{f^M})_3] d\tau \right| \quad (3.36)$$

and

$$K_{133}(\xi, t) \leq \left| \int_0^t [(\hat{T}_2(\xi, \tau) M_2 \widehat{f^M})_3 * (\hat{T}_2(\xi, \tau) M_2 \widehat{f^M})_3] d\tau \right|. \quad (3.37)$$

Similar to (3.27), we obtain

$$\|K_{131}\|_{L^1(E)} + \|K_{123}\|_{L^1(E)} + \|K_{133}\|_{L^1(E)} \leq C(t + t^2). \quad (3.38)$$

And similarly,

$$\|K_{113}\|_{L^1(E)} + \|K_{132}\|_{L^1(E)} \leq C(t + t^2). \quad (3.39)$$

Summing up (3.21), (3.28), (3.29), (3.34), (3.38) and (3.39), we get

$$\|K_1\|_{L^1(E)} \geq c - \frac{CN^4}{\Omega^4 M 2^{2M}} - C(t + t^2) \quad (3.40)$$

for $-\nu 2^{-2M} \leq t \leq \frac{1}{N}$ with $M \geq \frac{1}{2} \log_2 \frac{N}{\nu}$.

(ii) Estimates for $K_2(\xi, t)$ and $K_3(\xi, t)$ with $\xi \in E$

Noticing that

$$|\hat{T}_2(\zeta, \tau)| \leq \frac{|\zeta'|}{|\zeta|} \tau \leq Nt$$

and

$$|\hat{T}_3(\zeta, \tau) - \hat{T}_1(\zeta, \tau)| \leq 1 - \cos\left(\frac{|\zeta|'}{|\zeta|}\tau\right) \leq \frac{1}{2} \frac{|\zeta|'^2}{|\zeta|^2} \tau^2 \leq \frac{1}{2} N^2 t^2$$

for all $\tau \in (0, t)$ and all $\zeta \in \mathbb{R}^3$, it is easy to observe for $\xi \in E$ that

$$K_2(\xi, t) + K_3(\xi, t) \leq C(t + t^2) \int_0^t \int_{\mathbb{R}^3} e^{-\nu(|\xi-\eta|^2 + |\eta|^2)\tau} |\widehat{f^M}(\xi - \eta)| |\widehat{f^M}(\eta)| d\eta d\tau.$$

The same process for getting (3.18) results in

$$\|K_2 + K_3\|_{L^1(E)} \leq C(t + t^2). \quad (3.41)$$

Finally, summing up (3.40) and (3.41), we see that there exist $0 < c$ and $C > 1$ such that

$$\|A_2(f^M)(t)\|_{F_{1,r}^{-1}} \geq c - \frac{CN^4}{\Omega^4 M 2^M} - C(t + t^2)$$

for $\nu^{-1} 2^{-2M} \leq t \leq \frac{1}{N}$ with $M \geq \frac{1}{2} \log_2 \frac{N}{\nu}$. In fact, there holds

$$\|A_2(f^M)(t)\|_{F_{1,r}^{-1}} \geq \frac{c}{3}$$

for all $M \in \mathbb{N}$ and $t > 0$ with $M \geq \max\{\frac{3CN^4}{\Omega^4 c}, \frac{1}{2} \log_2 \frac{6C}{c\nu}, \frac{1}{2} \log_2 \frac{N}{\nu}, \frac{1}{2} \log_2 \frac{1}{\nu}\}$ and $\nu^{-1} 2^{-2M} \leq t \leq \min\{\frac{c}{6C}, \frac{1}{N}, 1\}$, which implies (3.3). This completes the proof of Theorem 1.6.

References

- [1] A. Babin, A. Mahalov and B. Nicolaenko, On the asymptotic regimes and the strongly stratified limit of rotating Boussinesq equations, *Journal of Theoretical and Comp. Fluid Dynamics*, 9(1997), 223-251.
- [2] A. Babin, A. Mahalov and B. Nicolaenko, On the regularity of three-dimensional rotating Euler-Boussinesq equations, *Math. Models Methods Appl. Sci.*, 9(1999), 1089-1121.
- [3] A. Babin, A. Mahalov and B. Nicolaenko, Fast singular oscillating limits and global regularity for the 3D primitive equations of geophysics, Special issue for R. Temam's 60th birthday, *M2AN Math. Model. Numer. Anal.* 34(2000), 201-222.
- [4] A. Babin, A. Mahalov and B. Nicolaenko, Strongly stratified limit of 3D primitive equations in an infinite layer, *Contemp. Math.*, 283, 2001.
- [5] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren Math. Wiss. vol. 343, Springer-Verlag, Berlin, Heidelberg, 2011.
- [6] I. Bejenaru and T. Tao, Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation, *J. Funct. Anal.* 233(2006), 228-259.
- [7] J. Bourgain and N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D, *J. Funct. Anal.* 255(2008), 2233-2247.
- [8] J.-M. Bony, Symbolic calculus and propagation of singularities for nonlinear partial differential equations(French), *Ann. Sci. École Norm. Sup.* 14(1981), 209-246.

- [9] M. Cannone, A generalization of a theorem by Kato on Navier-Stokes equations. *Rev. Mat. Iberoam.*, 13(1997), 515-541.
- [10] M. Cannone and G. Karch, Smooth or singular solutions to the Navier-Stokes system? *J. Differential Equations* 197(2004), 247-274.
- [11] M. Cannone and G. Wu, Global well-posedness for Navier-Stokes equations in critical Fourier-Herz spaces, *Nonlinear Anal.* 75(2012), 3754-3760.
- [12] C. Cao and E. S. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, *Ann. of Math.* 166(2007), 245-267.
- [13] C. Cao and E. S. Titi, Global well-posedness of the 3D primitive equations with partial vertical turbulence mixing heat diffusion, *Comm. Math. Phys.* 310(2012), 537-568.
- [14] C. Cao, J. Li and E. S. Titi, Global well-posedness for the 3D primitive equations with only horizontal viscosity and diffusion, *Commun. Pure Appl. Math.* (to appear)
- [15] F. Charve, Global well-posedness and asymptotics for a geophysical fluid system, *Comm. Partial Differential Equations* 29(2004), 1919-1940.
- [16] F. Charve, Convergence of weak solutions for the primitive system of the quasigeostrophic equations, *Asymptot. Anal.* 42(2005), 173-209.
- [17] F. Charve, Asymptotics and vortex patches for the quasigeostrophic approximation, *J. Math. Pures Appl.* 85(2006), 493-539.
- [18] F. Charve, Global well-posedness for the primitive equations with less regular initial data, *Ann. Fac. Sci. Toulouse Math.* 17(2008), 221-238.
- [19] F. Charve, Asymptotics and lower bound for the lifespan of solutions to the Primitive Equations, *arXiv:1411.6859*.
- [20] F. Charve and V. Ngo, Global existence for the primitive equations with small anisotropic viscosity, *Rev. Mat. Iberoam.* 27(2011), 1-38.
- [21] J.-Y. Chemin, À propos d'un problème de pénalisation de type antisymétrique(French), *J. Math. Pures Appl.* 76(1997), 739-755.
- [22] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Anisotropy and dispersion in rotating fluids, *Nonlinear PDEs and Applications*, Stud. Math. Appl. 31, North Holland, 2002.
- [23] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical geophysics. An introduction to rotating fluids and the Navier-Stokes equations*, Volume 32 of Oxford Lecture Series in Mathematics and its Applications. The Clarendon Press, Oxford University Press, Oxford, 2006.
- [24] S. Cui, Sharp well-posedness and ill-posedness of the Navier-Stokes initial value problem in Besov-type spaces, *arXiv:1505.00865*.
- [25] B. Cushman-Roisin, *Introduction to geophysical fluid dynamics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1994.
- [26] D. Fang, B. Han and M. Hieber, Global existence results for the Navier-Stokes equations in the rotational framework in Fourier-Besov spaces, in W. Arendt, R. Chill, Y. Tomilov(Eds.), *Operator Semigroups meet Complex Analysis, Harmonic Analysis and Mathematical Physics*, Birkhauser, to appear.

- [27] D. Fang, B. Han and M. Hieber, Local and global existence results for the Navier-Stokes equations in the rotational framework, *Commun. Pure Appl. Anal.* 14(2015), 609-622.
- [28] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Arch. Ration. Mech. Anal.* 16(1964), 269-315.
- [29] Y. Giga, Solutions of semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, *J. Differential Equations* 62(1986), 186-212.
- [30] Y. Giga and T. Miyakawa, Navier-Stokes flows in \mathbb{R}^3 with measure as initial vorticity and the Morrey spaces, *Comm. Partial Differential Equations* 14(1989), 577-618.
- [31] Y. Giga, K. Inui, A. Mahalov and J. Saal, Uniform global solvability of the rotating Navier-Stokes equations for nondecaying initial data, *Indiana Univ. Math. J.* 57(2008), 2775-2791.
- [32] M. Hieber and Y. Shibata, The Fujita-Kato approach to the Navier-Stokes equations in the rotational framework, *Math. Z.* 265(2010), 481-491.
- [33] T. Iwabuchi, Global well-posedness for Keller-Segel system in Besov type spaces, *J. Math. Anal. Appl.* 379(2011), 930-948.
- [34] T. Iwabuchi and R. Takada, Global solutions for the Navier-Stokes equations in the rotational framework, *Math. Ann.* 357(2013), 727-741.
- [35] T. Iwabuchi and R. Takada, Global well-posedness and ill-posedness for the Navier-Stokes equations with the Coriolis force in function spaces of Besov type, *J. Funct. Anal.* 267(2014), 1321-1337.
- [36] H. Koba, A. Mahalov and T. Yoneda, Global well-posedness for the rotating Navier-Stokes-Boussinesq equations with stratification effects, *Adv. Math. Sci. Appl.* 22(2012), 61-90.
- [37] T. Kato, Strong L^p -solutions of the Navier-Stokes equation in \mathbb{R}^m , with applications to weak solutions, *Math. Z.* 187(1984), 471-480.
- [38] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations, *Adv. Math.* 157(2001), 22-35.
- [39] Y. Koh, S. Lee and R. Takada, Dispersive estimates for the Navier-Stokes equations in the rotational framework, *Adv. Differential Equations* 19(2014), 857-878.
- [40] P. Konieczny and T. Yoneda, On dispersive effect of the Coriolis force for the stationary Navier-Stokes equations, *J. Differential Equations* 250(2011), 3859-3873.
- [41] Z. Lei and F. Lin, Global mild solutions of Navier-Stokes equations, *Comm. Pure Appl. Math.* 64(2011), 1297-1304.
- [42] Q. Liu and J. Zhao, Global well-posedness for the generalized magneto-hydrodynamic equations in the critical Fourier-Herz spaces, *J. Math. Anal. Appl.* 420(2014), 1301-1315.
- [43] J. Pedlosky, *Geophysical fluid dynamics*, 2nd edn. Springer-Verlag, New York, 1987.
- [44] F. Planchon, Solutions globales et comportement asymptotique pour les équations de Navier-Stokes, Thèse, Ecole Polytechnique, 1996.
- [45] R. Salmon, *Lectures on geophysical fluid dynamics*, Oxford University Press, New York, 1998.

- [46] J. Sun, M. Yang and S. Cui, Existence and analyticity of mild solutions for the 3D rotating Navier-Stokes equations(*submitted*).
- [47] B. Wang, Ill-posedness for the Navier-Stokes equations in critical Besov spaces $\dot{B}_{\infty,q}^s$. *Adv. Math.* 268(2015), 350-372.
- [48] T. Yoneda, Ill-posedness of the 3D-Navier-Stokes equations in a generalized Besov space near BMO^{-1} , *J. Funct. Anal.* 258(2010), 3376-3387.